

# Universal Quantum Computation by Holonomic and Non-Local Gates with Imperfections

Demosthenes Ellinas<sup>1</sup> \* and Jiannis Pachos<sup>2,3</sup> †

<sup>1</sup> *Technical Univeristy of Crete, Department of Sciences, Section of Mathematics GR - 731 00 Chania, Crete Greece*

<sup>2</sup> *Max Planck Institut für Quantenoptik, D-85748 Garching, Germany<sup>‡</sup>*

<sup>3</sup> *Institute of Scientific Interchange Foundation, Villa Gualino, Viale Settimio Severo 65, I-10133 Torino, Italy*  
(February 1, 2008)

We present a non-local construction of universal gates by means of holonomic (geometric) quantum teleportation. The effect of the errors from an imperfect control of the classical parameters, the looping variation of which builds up holonomic gates, is investigated. Additionally, the influence of quantum decoherence on holonomic teleportation used as a computational primitive is studied. Advantages of the holonomic implementation with respect to control errors and dissipation are presented.

PACS numbers: 03.67.Lx, 03.67.Hk

Holonomic quantum computation (*HQC*) is a mathematically fascinating area where elements from differential geometry are employed in order to describe the logical evolution of a quantum system with (multiple) degenerate energy eigen-states [1–4]. Recent works [5–7] support the belief that holonomic implementation in NMR, quantum optics or ion traps may be a possible avenue for quantum computation. Here our aim is twofold: first, the scope of *HQC* is extended to the general framework of universal quantum computation with the construction of non-local gates; second, non ideal *HQC* implementations, in the form of geometrical imperfections of adiabatic parametric closed paths needed for the construction of holonomic gates, as well as the effect of the simultaneous presence of decoherence, is formulated and investigated. Our choice for basing the whole construction on the error-avoiding paradigm of holonomic quantum wiring stems from the fact that due to the geometric nature of making the irreducible connections corresponding to holonomic gates, the latter are expected to be resilient to certain types of errors. As this work shows, under conditions specified below this turns out to be true.

Specifically, the methodology for the generation of holonomies is as follows. In the control parametric manifold of iso-spectral transformations of a given degenerate Hamiltonian closed paths (loops) are run adiabatically in order to represent the evolution operation for each degenerate eigenspace as a holonomy of a given connection,  $A$  [8–10]. The latter has a form defined from the structure of the bundle of the energy degenerate spaces [11] and its irreducibility manifests the universality of the holonomic gates. The loops, when subject to imperfec-

tions while spanned, introduce an error in the final gates through their accordingly fluctuated parameters. This is systematically studied for the Hadamard (H) and the control-not (CN) gates, which are used to build up a teleportation circuit robust to control errors. Also the H and CN gates are constructed non locally by using the teleportation circuit as a building primitive. The choice for implementing only these two gates is based on the following fact; although H and CN belong in the Pauli group  $C_1$  and the Clifford group  $C_2$  respectively, and need to be supplemented by an element of the class of gates  $C_3 \equiv \{U/UC_1U^\dagger \subseteq C_2\}$  such as the  $T$  Toffoli gate, the  $\pi/8$  gate (rotation about the  $z$ -axis by an angle  $\pi/4$ ), or the controlled-phase gate ( $\text{diag}(1, 1, 1, i)$ ), in order to achieve universality [12,13], some of them are proven to be constructable fault-tolerantly in circuits involving only measurements of  $C_2$  gates (see e.g. [14] for the  $T$  gate, and [15] for similar construction of the other gates). In this way we can obtain universality by studying the H and CN gates only. As a figure of merit of the effect of both geometric imperfections and quantum dissipation on the overall performance of the teleportation circuit its fidelity is investigated for various limiting values of the decoherence and the imperfection parameters.

As a starting point we shall use the  $\mathbf{CP}^n$  holonomic model [11] in order to acquire the desired quantum gates [16]. In [11] a mathematical way for the holonomic construction of given gates by running specific loops in the control parametric manifold is presented. The initial Hamiltonian of this model is given by  $H_0 = \varepsilon_0|n+1\rangle\langle n+1|$  which acts on the state-space spanned by  $\{|\alpha\rangle\}_{\alpha=1}^{n+1}$ . We assume that it is possible by external con-

---

\*Electronic address: ellinas@science.tuc.gr

†Electronic address: jip@mpq.mpg.de

‡Present Address

trol to perform equivalent transformations of  $H_0$  given by  $\mathcal{O}(H_0) := \{\mathcal{U} H_0 \mathcal{U}^\dagger / \mathcal{U} \in U(n+1)\}$ . Their parametric space is isomorphic to the  $n$ -dimensional complex projective space  $\mathbf{CP}^n \cong U(n+1)/(U(n) \times U(1))$  which is at the disposal of the experimenter. Each point,  $\mathbf{z}$ , of the  $2n$  dimensional  $\mathbf{CP}^n$  manifold corresponds to a unitary matrix  $\mathcal{U}(\mathbf{z}) = U_1(z_1)U_2(z_2)...U_n(z_n)$ , where  $U_\alpha(z_\alpha) = \exp[G_\alpha(z_\alpha)]$  with  $G_\alpha(z_\alpha) = z_\alpha|\alpha\rangle\langle n+1| - \bar{z}_\alpha|n+1\rangle\langle\alpha|$  and  $z_\alpha = \theta_\alpha e^{i\phi_\alpha}$ , for  $\alpha = 1, \dots, n$ . If  $|\psi\rangle_{in}$  is the initial state in the zero degenerate space of the Hamiltonian, at the end of the adiabatic run of a loop  $C$  in the control manifold  $\mathbf{CP}^n$  one obtains  $|\psi\rangle_{out} = \Gamma_A(C) |\psi\rangle_{in}$ . The holonomy  $\Gamma_A(C) \in U(n)$  has a geometric origin and its appearance accounts for the non-trivial curvature of the bundle of eigenspaces over  $\mathbf{CP}^n$ . By introducing the Wilczek-Zee connection [9]  $A_{\alpha\bar{\alpha}}^\mu := \langle \bar{\alpha} | \mathcal{U}^\dagger(\mathbf{z}) \frac{\partial}{\partial z_\mu} \mathcal{U}(\mathbf{z}) | \alpha \rangle$ , with  $\alpha, \bar{\alpha} = 1, \dots, n$ , one finds  $\Gamma_A(C) = \mathbf{P} \exp \int_C A$  [8], where  $\mathbf{P}$  denotes path ordering.

For particular loops the following holonomies are calculated [11]; for the loop  $C_1 \in (\theta_\beta, \phi_\beta)$ , we obtain an abelian like holonomy,  $\Gamma_A(C_1) = e^{-i\Sigma_1} |\beta\rangle\langle\beta| + |\beta^\perp\rangle\langle\beta^\perp|$ , where  $\mathcal{H} = \text{span}\{|\beta\rangle, |\beta^\perp\rangle\}$ . The area  $\Sigma_1 = \int_{D(C_1)} d\theta_\beta d\phi_\beta \cos\theta_\beta$  may be represented as the one of the surface enclosed by  $C_1$  on a  $S^2$  sphere with coordinates  $(2\theta_\beta, \phi_\beta)$ , while  $D(C_1)$  is the enclosed surface on the  $(\theta_\beta, \phi_\beta)$  plane. For  $C_2 \in (\theta_\beta, \phi_{\bar{\beta}})$ ,  $\bar{\beta} > \beta$ , we take  $\Gamma_A(C_2) = e^{i\Sigma_2} |\bar{\beta}\rangle\langle\bar{\beta}| + |\bar{\beta}^\perp\rangle\langle\bar{\beta}^\perp|$  which is of similar abelian nature as  $\Gamma_A(C_1)$ . In order to obtain a non-abelian holonomy we perform the loop  $C_3$  on the plane  $(\theta_\beta, \theta_{\bar{\beta}})$  positioned at  $\phi_\beta = \phi_{\bar{\beta}} = 0$ , resulting to  $\Gamma_A(C_3) = \exp[-i(-i|\beta\rangle\langle\bar{\beta}| + i|\bar{\beta}\rangle\langle\beta|)\tilde{\Sigma}_3]$ , while by taking the plane  $(\theta_\beta, \phi_{\bar{\beta}})$  to be at the position  $\phi_\beta = \pi/2$  and  $\phi_{\bar{\beta}} = 0$ , we obtain  $\Gamma_A(C_4) = \exp[-i(|\beta\rangle\langle\bar{\beta}| + |\bar{\beta}\rangle\langle\beta|)\tilde{\Sigma}_4]$  where  $\tilde{\Sigma} = \int_{D(C)} d\theta_\beta d\theta_{\bar{\beta}} \cos\theta_{\bar{\beta}}$ . The identity action on the rest of the states is implied.

With these control manipulations holonomies are produced, which can be used as logical gates with parameters the areas  $\Sigma$ . In fact we obtain a whole set of closed paths in the parametric manifold which give the same holonomies, as deformations of the loop shape and position give the same gate provided their enclosed area is preserved. It is worth noticing that the composition rules [1] of loops of multiplied holonomies may reduce the total length of the transversed paths by combining loops on the same or perpendicular planes for successive gates eventually reducing the required resources for the overall circuit.

*Imperfect holonomies.* In order to study the errors introduced by imperfect control of the external parameters we adopt an imperfectly spanned loop,  $C'$ . If the errors are statistical rather than systematic then the area spanned by this loop are, to the first order, zero. Let us consider how systematic errors in the area effect

one and two qubit gates. The Hadamard gate is given by  $U_H = \begin{bmatrix} \cos \Sigma & \sin \Sigma \\ \sin \Sigma & -\cos \Sigma \end{bmatrix}$ , for  $\Sigma = \pi/4$ . Up to a corrective phase given by  $\Gamma_A(C_1)$  with  $\Sigma_1 = \pi$  it may be produced by a loop  $C_3$ , with spanning area given by  $\Sigma = \int_{D(C_3)} d\theta_1 d\theta_2 \cos\theta_1$  with  $D(C_3)$  taken to be a rectangular surface enclosed by  $\{0 \leq \theta_1 \leq \pi/2, 0 \leq \theta_2 \leq \pi/4\}$ . Introduce an error in this surface by translating the borders of  $\theta_1$  and  $\theta_2$  by  $\alpha$  and  $\beta$  respectively, where  $\alpha, \beta \ll 1$ . This is a kind of systematic error. The imperfect Hadamard gate is given to the first order in  $\varepsilon$  by  $U_H(\varepsilon) = U_H + \varepsilon h$ , with  $h = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ . The integrand  $\cos\theta_1$  in  $\Sigma$  is the  $(\theta_1, \theta_2)$  dependent part of the component of the field strength  $F_{\theta_1\theta_2} = \partial_{\theta_1} A_{\theta_2} - \partial_{\theta_2} A_{\theta_1} + [A_{\theta_1}, A_{\theta_2}]$ . As another interpretation of the holonomy, within this approach, is the exponential of the flux of  $F$  [17], then we want this flux to be stable with respect to small deformations of the relevant surface. Hence, we can take this surface to be such that fluctuations of its area give insignificant variations to the total flux. Indeed, the flux enclosed by the deformed loop  $C'_3$  is given by  $\Sigma(\varepsilon) \approx \frac{\pi}{4} + \varepsilon$ , times the Pauli matrix  $\sigma_2$ , with infinitesimal deviation  $\varepsilon = \beta$ , where the infinitesimal  $\alpha$  does not appear at all in the first order due to the choice of the position of the rectangular's sides. Ideally we would like  $F_{\theta_1\theta_2}$  to be exponentially decreasing with respect to the distance from a particular point of the control manifold so that for large loops centered at that point local deformations of the loop shape would not alter the enclosed flux. A model with such characteristics may be build with optical devices. Indeed, in [6] the one qubit gates  $\Gamma_A(C_I) = \exp[-i\hat{\sigma}_1 \Sigma_I]$  with  $C_I \in (x, r_1)_{\theta_1=0}$  and  $\Sigma_I := \int_{\Sigma(C_I)} dx dr_1 2e^{-2r_1}$  as well as  $\Gamma_A(C_{II}) = \exp[-i\hat{\sigma}_2 \Sigma_{II}]$  with  $C_{II} \in (y, r_1)_{\theta_1=\pi}$  and  $\Sigma_{II} := \int_{\Sigma(C_{II})} dy dr_1 2e^{-2r_1}$ , which can produce any one qubit operation, give for large values of the squeezing parameter  $r_1$  zero error in all orders of the loop deformation along  $r_1$ . This can be considered as an initial point for passing from geometrical QC to topological QC [18,19]. Further study is needed for the construction of optical (bosonic) two qubit gates with topological character.

On the other hand, the control-not gate is given, up to phase corrections again by a  $C_3$  loop between the proper  $(\theta_\beta, \theta_{\bar{\beta}})$  variables with  $\Sigma = \pi/2$ . By inserting the errors  $(\alpha', \beta')$  in the corresponding components the area becomes  $\Sigma(\delta) \approx \frac{\pi}{2} + \delta$ , with  $\delta = \beta' \ll 1$  the error deviation. Hence,  $U_{CN}(\delta) = U_{CN} - i\delta|1\rangle\langle 1| \otimes \mathbf{1}$ .

*Teleportation Circuit.* Consider the teleportation circuit [21] of Fig. 1.

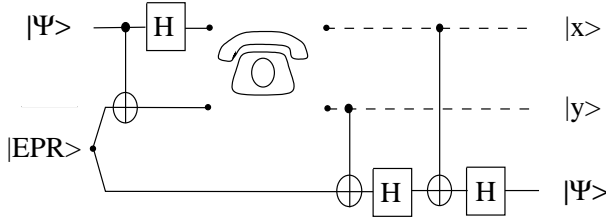


FIG. 1. The Brassard et.al. teleportation circuit. Dashed lines represent classical channels.

Depicted are the unknown state  $|\Psi\rangle = a_0|0\rangle + a_1|1\rangle$  which we wish to teleport, the initially employed EPR states  $|EPR\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ , while the classical states  $|x\rangle$  or  $|y\rangle$  are the outcomes of measurement in the middle of the circuit. For a more realistic implementation we may use the imperfect Hadamard and CN gates as presented above, where we can take perturbatively into account the imperfections of the Alice and Bob holonomies in order to estimate the reduction of the optimal fidelity of the above scheme, due to imperfect holonomic implementation. The overall circuit is represented as  $\mathcal{U}_{tel}(\varepsilon, \delta) = \mathcal{U}_6(\varepsilon)\mathcal{U}_5(\delta)\mathcal{U}_4(\varepsilon)\mathcal{U}_3(\delta)\mathcal{U}_2(\varepsilon)\mathcal{U}_1(\delta)$ , where,  $\varepsilon$  is the error for the Hadamard gate and  $\delta$  is the error for the CN gate. From the six gates presented in Fig. 1, the first CN gate is written as

$$\mathcal{U}_1(\delta) = U_{CN}(\delta) \otimes \mathbf{1} \approx U_{CN} \otimes \mathbf{1} - i\delta|1\rangle\langle 1| \otimes \mathbf{1} \otimes \mathbf{1}$$

$$\equiv \mathcal{U}_1 + \delta V_1$$

which is a unitary matrix up to order  $\mathcal{O}(\varepsilon)$ . The rest CN's are written similarly. The first Hadamard gate is given by

$$\mathcal{U}_2(\varepsilon) = U_H(\varepsilon) \otimes \mathbf{1} \otimes \mathbf{1} \approx U_H \otimes \mathbf{1} \otimes \mathbf{1} + \varepsilon V_2$$

$$\equiv \mathcal{U}_2 + \varepsilon V_2$$

and equivalently for  $\mathcal{U}_4(\varepsilon)$  and  $\mathcal{U}_6(\varepsilon)$ . The overall circuit, up to the first order in the error  $\varepsilon$  or  $\delta$  is given by  $\mathcal{U}_{tel}(\delta, \varepsilon) = \mathcal{U}_{tel} + \delta(V_1 + V_3 + V_5) + \varepsilon(V_2 + V_4 + V_6) \equiv \mathcal{U}_{tel} + \delta V_\delta + \varepsilon V_\varepsilon$ . To quantify the error of the teleported state due to the imperfections in the loop spanning we introduce the fidelity

$$\mathcal{F}_{\delta, \varepsilon}^{xy} = \min_{|\Psi\rangle} |\langle xy|\Psi|\mathcal{U}_{tel}(\delta, \varepsilon)|\Psi EPR\rangle|^2$$

where  $|xy\rangle$  is one of the possible outcomes  $|00\rangle, |01\rangle, |10\rangle$  or  $|11\rangle$  for the two first qubits due to measurement. In particular we find after minimization and tracing the different possibilities of  $|xy\rangle$  the result

$$\mathcal{F}_{\delta, \varepsilon} = 1 - \varepsilon \frac{3}{2}(\sqrt{2} - 1) - \delta \frac{1}{2\sqrt{2}} \quad (1)$$

which is smaller or equal to identity for small positive values of  $\varepsilon$  and  $\delta$ . Note that the coefficient of the CN's

error is almost *half* of the one of the H gates, which is advantageous as for the two qubit gates you need controllability of two qubit manipulations which should double  $\delta$  with respect to  $\varepsilon$ .

*Application.* Let us proceed by adopting the teleportation as a kind of computation primitive, which can accept a proper input state (quantum software) in a given site and provide output in another site a universal set of quantum gates, in a fault-tolerant way. The teleportation method for gate construction is adopted here as it essentially reduces the needed resources for quantum computation to special ancilla state preparation and also provides unifying technique for building in a fault-tolerant way a hierarchy of quantum gates [12,15,22].

Our aim is to use the Brassard's et.al. circuit of teleportation in order to produce (teleport) H and CN gates, which can be initially implemented on the EPR states fault-tolerantly. For that we consider imperfect holonomic realization of these generalized circuits and within first order approximation, evaluate their robustness by obtaining their fidelity. Due to the similarity of the operators involved the derived fidelities are closely related with the one in (1). Indeed, for the Hadamard gate the employed circuit is  $U_{tel}^H = (\mathbf{1} \otimes \mathbf{1} \otimes U_H) U_{tel} (\mathbf{1} \otimes \mathbf{1} \otimes U_H^\dagger)$  and its fidelity  $\mathcal{F}_H$  with respect to the transformed initial and final states, as can easily be shown, is equal to the fidelity of the teleportation circuit itself, i.e  $\mathcal{F}_H = \mathcal{F}$ . For constructing the CN teleported gate we shall employ two teleportation circuits with the additional permutation operator  $\Pi_{13} = \sum_{x,y=0,1} |y\rangle\langle x| \otimes \mathbf{1} \otimes |x\rangle\langle y|$ . The circuit of the two teleportations (see Fig. 2) is arranged as  $W_{tel} = U_{tel} \otimes \Pi_{13} U_{tel} \Pi_{13}$  and the CN implementation is given by  $W_{tel}^{CN} \equiv U_{CN}^{34} W_{tel} U_{CN}^{\dagger}$ . By calculating the fidelity of this circuit with respect to the initial and final states we obtain  $\mathcal{F}_{CN}(\varepsilon, \delta) = \mathcal{F}(2\varepsilon, 2\delta)$ ; that is, it has the same functional form as in the case of one teleportation but the errors are now doubled. These results are valid to all orders in  $\varepsilon$  and  $\delta$ .

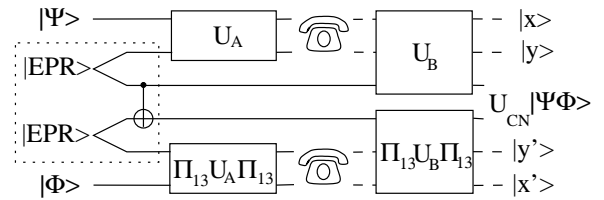


FIG. 2. The CN teleported gate. The dotted box includes the initial state sub-circuit, which can be implemented fault tolerantly.

*Dissipation.* Let us now consider the case where imperfections in the construction of holonomic gates as have been studied so far are present simultaneously with dissipative mechanisms in the modeling of the circuits. As dissipation is an almost unavoidable destruction of quantum coherence that affects the performance quality of

logical circuits, it is expected to cooperate with the possible imperfections in lowering their fidelity. To quantify these thoughts we shall formulate the appearance of a general class of dissipative mechanisms in the computational primitive element of an imperfect teleportation circuit. More specifically we shall study decoherence on the Bob's part of the total density operator of the teleportation scheme, that takes place after the completion of Alice's part of the circuit and during the time she classically transmits two bits of information to Bob. This can be also thought of as an imperfection during the measurement procedure in the middle of the circuit. Let  $\rho_I = |\Psi\rangle\langle\Psi| \otimes \rho$ , the initial density operator with  $|\Psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$  the state to be transmitted, and  $\rho$  the transmitting density operator, which in general can be taken not to be a perfect projector of an *EPR* entangled pair. Let  $\overline{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ , the total Hilbert space of Alice and Bob, where  $\mathcal{H} = \text{span}\{|0\rangle, |1\rangle\}$ . Then let  $\mathcal{P} = \text{End}(\overline{\mathcal{H}})$ , the space of pure density operators acting on  $\overline{\mathcal{H}}$ , and  $\mathcal{S} = \text{hull}(\mathcal{P})$ , the convex hull of  $\mathcal{P}$ . Consider the linear, trace preserving and completely positive family of maps  $\{s_\lambda : \mathcal{S} \rightarrow \mathcal{S} : \lambda \geq 0\}$ , that admits a Kraus operator-sum representation such that  $s_\lambda(\rho) = \sum_{i=1}^k W_i \rho W_i^\dagger$ , where  $\{W_i\}_{i=1}^k \in \text{End}(\overline{\mathcal{H}})$  and  $\sum_{i=1}^k W_i^\dagger W_i = \mathbf{1}$ . Since we are interested in dissipation occurring in Bob's site only, we take  $W_i = \mathbf{1} \otimes \mathbf{1} \otimes V_i$ , for some chosen  $V_i$ 's. Moreover, the dissipation generators  $W_i(\lambda)$  may be taken to depend on the parameter  $\lambda$ , in such a way that  $\lim_{\lambda \rightarrow 0} W_1(\lambda) = \mathbf{1}$ ,  $\lim_{\lambda \rightarrow 0} W_i(\lambda) = \mathbf{0}$ ,  $i \neq 1$ , namely in the zero dissipation limit  $s_{\lambda=0}(\rho) = \rho$ . Then we rewrite  $s_\lambda(\rho) = \sum_{i=1}^k \text{Ad}(W_i)\rho$ , where the adjoint action  $\text{Ad}(X)\rho \equiv X\rho X^\dagger$  is employed. By means of the property  $\text{Ad}(XY) = \text{Ad}(X)\text{Ad}(Y)$ , we now introduce a 3-parameter POVM  $\{\mu_{\delta,\varepsilon,\lambda} : \mathcal{S} \rightarrow \mathcal{S} : 0 \leq \delta \leq 1, 0 \leq \varepsilon \leq 1, \lambda \geq 0\}$ , where

$$\mu_{\delta,\varepsilon,\lambda}(\rho_I) = \sum_{i=1}^k \text{Ad}(\mathcal{U}_{Bob}(\delta,\varepsilon)) \text{Ad}(W_i(\lambda)) \text{Ad}(\mathcal{U}_{Alice}(\delta,\varepsilon)) \rho_I.$$

As previously the unitary operators implementing the gates of Alice and Bob in the teleportation circuit are parameterized by the imperfection parameters  $\delta, \varepsilon$ . Then we observe that the dissipation operator on Bob's site commutes with Alice unitary operation i.e.,  $[\mathcal{U}_{Alice}, W_i] = 0$ ,  $i = \{1, \dots, k\}$ , so we have that

$$\mu_{\delta,\varepsilon,\lambda}(\rho_I) = \sum_{i=1}^k \text{Ad}(\mathcal{U}_{tel}(\delta,\varepsilon)) \text{Ad}(W_i(\lambda)) \rho_I = \mathcal{U}_{tel}(\delta,\varepsilon) s_\lambda(\rho_I) \mathcal{U}_{tel}^\dagger(\delta,\varepsilon).$$

At this point there are two ways to proceed. The first one is based on the observation that the above dissipative teleportation scheme is equivalent to the teleportation scheme in which Alice and Bob share a mixed

entangled state and an enhancement of the quantum teleportation fidelity is achieved by allowing either of them, to initially perform a local dissipative interaction with the environment [23]. Specifically let  $V(\cdot) = \sum_{i=1}^k V_i(\cdot) V_i^\dagger$ , then the closeness of the initially shared bipartite state  $\mathbf{1} \otimes V(\rho)$ , to the ideal maximally entangled state  $P_{EPR} = |EPR\rangle\langle EPR|$ , is quantified by the *fully entangled fraction* [24], of the bipartite state,  $f = \max_V \text{Tr}(\mathbf{1} \otimes V(\rho) P_{EPR})$ . According to the analysis of [23], we search for such  $V$  and  $\rho$  that  $f > 1/2$ , so that the optimal fidelity of the teleportation  $\mathcal{F} = \frac{2f+1}{3}$ , exceeds the limit of the classical communication viz.  $\mathcal{F}_{cl} = \frac{2}{3}$ .

Alternatively, we can simply proceed by assuming that our dissipative holonomic teleportation has a lower fidelity compared to the ideal teleportation scheme and perform a first order perturbation say, of the holonomic parameters  $\delta, \varepsilon$  in order to estimate how close to one our fidelity can be. Let us take the latter possibility and choose for definiteness the phase damping mechanism described by the  $k = 2$  POVM, with  $V_1 = \text{diag}(1, e^{-\lambda})$ , and  $V_2 = \text{diag}(0, \sqrt{1 - e^{-2\lambda}})$ . In terms of projectors  $P_{ab} \equiv |a\rangle\langle b|$ , the initial state is written as  $\rho_I^\Psi = |\Psi\rangle\langle\Psi| \otimes P_{EPR} = \frac{1}{2} \sum_{i,j \in (0,1)} \alpha_i \overline{\alpha}_j P_{ij} \otimes P_{EPR}$ . Similarly if Alice measurement results in two classical bits  $(x, y)$ , then the final state of teleportation is  $\rho_F^\Psi = |xy\rangle\langle xy| \otimes P_{xy} = \frac{1}{2} \sum_{k,l \in (0,1)} \alpha_k \overline{\alpha}_l P_{xx} \otimes P_{yy} \otimes P_{kl}$ . To estimate the quality of the dissipative holonomic teleportation scheme against the standard ideal teleportation we shall use the previously introduced fidelity factor in its equivalent Hilbert-Schmidt or trace-norm form [13], which compares general mixed-state density operators and for our case reads:

$$\mathcal{F}_{\delta,\varepsilon,\lambda}^{x,y} = \min_{|\Psi\rangle} \text{Tr}(\mu_{\delta,\varepsilon,\lambda}(\rho_I^\Psi) \rho_F^\Psi).$$

After adding up all the different possibilities of  $x$  and  $y$  we obtain for the fidelity up to the first order in  $\varepsilon$  and  $\delta$ , but to all orders in  $\lambda$  the following expression

$$\begin{aligned} \mathcal{F}_{\delta,\varepsilon,\lambda} &= \frac{1}{2} (1 + e^{-\lambda}) - \\ &\varepsilon \frac{3}{2} (\sqrt{2} - 1) + \varepsilon (1 - e^{-\lambda}) \left( \frac{21}{32} \sqrt{7} - \frac{51}{32} \right) - \\ &\delta \frac{1}{2\sqrt{2}} + \delta (1 - e^{-\lambda}) \frac{3}{16} \sqrt{\frac{3}{2}}. \end{aligned} \quad (2)$$

The intriguing characteristic is that after allowing for dissipation to occur in the initial state by having non-zero values for  $\lambda$  the coefficients of  $\varepsilon$  and  $\delta$  become smaller, compared to the dissipationless case of eq.(1). Analytically, let  $\Delta\mathcal{F}_{\varepsilon,\delta} = \frac{1}{2} - \varepsilon \frac{1}{32} (21\sqrt{7} - 51) - \delta \frac{1}{4} \sqrt{\frac{3}{2}}$ , then expression (2) takes the form  $\mathcal{F}_{\delta,\varepsilon,\lambda} = \mathcal{F}_{\varepsilon,\delta} - (1 - e^{-\lambda}) \Delta\mathcal{F}_{\varepsilon,\delta}$ .

Clearly the initial value  $\mathcal{F}_{\varepsilon,\delta}$ , of the fidelity for zero dissipation  $\lambda = 0$ , changes to the asymptotic non-zero value  $\mathcal{F}_{\delta,\varepsilon,\infty} = \mathcal{F}_{\varepsilon,\delta} - \Delta\mathcal{F}_{\varepsilon,\delta}$ , for large dissipation  $\lambda \rightarrow \infty$ . This signifies the fact that the fidelity of imperfect holonomic teleportation becomes resilient to some quantum dissipation that may occur during classical transmission of information.

*Conclusions.* In this work holonomic H and CN gates are employed for the construction of the teleportation circuit, which provides itself as an architectural unit for building in a distributive and fault-tolerant way remote gates that effectively form a universal set for quantum computation, by means of a procedure that requires only prior searing of ancillary states between remote parties and quantum measurements. Although such a pivotal holonomic circuit could be considered protected from quantum errors due to the geometrical functioning principle of its holonomic gates, possible errors coming from systematic geometrical imperfections of the construction of holonomies, as well as from the presence of quantum noise during classical communication between remote parties of the circuit, are almost unavoidable. The modeling and studying of these types of errors showed that the holonomic teleportation circuit functioning under small imperfections of its gates is resilient to quantum noise during classical transmission.

Experimentally the theoretical construction presented here may be materialized with ion traps. Such an implementation of HQC, which uses the ion vibronic modes for the control manipulations presented in [6] and enjoys exponential dumping in control errors, is currently under investigation [20]. Furthermore, the teleportation scheme with ion traps presented in [25] may be performed with holonomic gates materializing practically the second part of our proposal and hence exhibiting the resilience to both the control and quantum noise errors.

JP would like to thank Christof Zalka for useful conversations. JP acknowledges a TMR Network support under the contract no. ERBFMRXCT96-0087.

---

[1] P. Zanardi and M. Rasetti, Phys. Lett. A **264** (1999) 94, quant-ph/9904011.  
[2] J. Pachos and P. Zanardi, "Quantum Holonomies for Quantum Computing", quant-ph/0007110.  
[3] J. Pachos, "Quantum Computation by Geometrical Means", to be published in the AMS Contemporary Math Series volume entitled "Quantum Computation and Quantum Information Science", quant-ph/0003150.  
[4] E. Sjöqvist, A. Pati, A. Ekert, J. Anandan, M. Ericsson, D. Oi and V. Vedral, Phys. Rev. Lett. **85** (2000) 2845.  
[5] A. Ekert, M. Ericsson, P. Hayden, H. Inamori, J. A.

Jones, D. K. L. Oi and V. Vedral, "Geometric Quantum Computation", quant-ph/0004015; J. A. Jones, V. Vedral, A. Ekert and G. Castagnoli, "Geometric quantum computation with NMR", Nature 403 869-871 (2000), quant-ph/9910052.  
[6] J. Pachos and S. Chountasis, Phys. Rev. A **62**, 052318 (2000), quant-ph/9912093.  
[7] I. Fuentes-Guridi, S. Bose, V. Vedral, "Proposal for measurement of harmonic oscillator Berry phase in ion traps", quant-ph/0006112.  
[8] For a review see, *Geometric Phases in Physics*, A. Shapere and F. Wilczek, Eds. World Scientific, 1989  
[9] F. Wilczek and A. Zee., Phys. Rev. Lett. **52**, 2111 (1984)  
[10] M. Nakahara, *Geometry, Topology and Physics*, IOP Publishing Ltd., 1990.  
[11] J. Pachos, P. Zanardi and M. Rasetti, Phys. Rev. A **61** 010305(R), quant-ph/9907103.  
[12] D. Gottesman, in *Group 22: Proc. of the XXII Inter. Symp. on Group Theor. Methods in Physics*, eds. S. P. Corney, R. Delbourgo and P. D. Jarvis, pp. 32 (Cambridge, MA International Press), quant-ph/9807006.  
[13] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information*, (Cambridge University Press, Cambridge, 2000).  
[14] P. W. Shor, Proc. 35th Annual Symp. on Fundamentals of Computer Science (IEEE Press, Los Alamos, 1996) p. 56 (quant-ph/9605011).  
[15] X. Zhou, D. W. Leung and I. L. Chuang, quant-ph/0002039.  
[16] The tensor product structure, which is missing from the  $\mathbf{CP}^n$  model [11] does not affect the results presented here on imperfections as well as the overall error of the circuits. In the holonomic context the complexity may be faced successfully as in [6] and [3], while the  $\mathbf{CP}^n$  model is simple enough for presenting a complete study of control imperfections.  
[17] The Abelianization of the theory by restricting on planes with commuting connection components eliminates the path ordering symbol and the treatment is as in the Abelian case.  
[18] A. Yu. Kitaev, "Fault-tolerant quantum computation by anyons", quant-ph/9707021.  
[19] J. Preskill, *Fault-tolerant quantum computation* in *Introduction to quantum computation and information*, Hoi-Kwong Lo, S. Popescu and T. Spiller Eds., World Scientific, Singapore, 1999.  
[20] Paper in preparation.  
[21] G. Brassard, S. Braunstein, R. Cleve, Physica D **120**, 43 (1998).  
[22] M. A. Nielsen and I. L. Chuang, Phys. Rev. Lett. **79** 321 (1997); D. Gottesman and I. L. Chuang, Nature **402** 390 (1999).  
[23] P. Badziag, M. Horodecki, P. Horodecki and R. Horodecki, arXiv:quant-ph/9912098.  
[24] C. H. Bennett, D. P. Di Vincenzo, J. Smolin, and W. K. Wothers, Phys. Rev. **54** 3814 (1997).  
[25] E. Solano, C. Cesar, R. de Matos Filho and N. Zagury, "Reliable teleportation in trapped ions", quant-ph/9903029.